

# STABILITY OF GORENSTEIN FLAT CATEGORIES WITH RESPECT TO A SEMIDUALIZING MODULE \*

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**Abstract:** In this paper, we first introduce  $\mathcal{W}_F$ -Gorenstein modules to establish the following Foxby equivalence:

$$\mathcal{G}(\mathcal{F}) \cap \mathcal{A}_C(R) \begin{array}{c} \xrightarrow{\quad C \otimes_R - \quad} \\ \xleftarrow{\quad \text{Hom}_R(C, -) \quad} \end{array} \mathcal{G}(\mathcal{W}_F)$$

where  $\mathcal{G}(\mathcal{F})$ ,  $\mathcal{A}_C(R)$  and  $\mathcal{G}(\mathcal{W}_F)$  denote the class of Gorenstein flat modules, the Auslander class and the class of  $\mathcal{W}_F$ -Gorenstein modules respectively. Then, we investigate two-degree  $\mathcal{W}_F$ -Gorenstein modules. An  $R$ -module  $M$  is said to be two-degree  $\mathcal{W}_F$ -Gorenstein if there exists an exact sequence

$$\mathbb{G}_\bullet = \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

in  $\mathcal{G}(\mathcal{W}_F)$  such that  $M \cong \text{im}(G_0 \rightarrow G^0)$  and that  $\mathbb{G}_\bullet$  is  $\text{Hom}_R(\mathcal{G}(\mathcal{W}_F), -)$  and  $\mathcal{G}(\mathcal{W}_F)^+ \otimes_R -$  exact. We show that two notions of the two-degree  $\mathcal{W}_F$ -Gorenstein and the  $\mathcal{W}_F$ -Gorenstein modules coincide when  $R$  is a commutative GF-closed ring.

**Keywords:** Semidualizing module;  $G_C$ -flat module;  $\mathcal{W}_F$ -Gorenstein module; Auslander class; Bass class; Stability of category.

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## 1. Introduction

Throughout this article,  $R$  is a commutative ring with identity and all modules are unitary. We denote by  $R\text{-Mod}$  the category of  $R$ -modules. For an  $R$ -module  $M$ , the Pontryagin dual or character module  $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$  is denoted by  $M^+$ .

Recall from White [16] that an  $R$ -module  $C$  is said to be semidualizing if  $C$  admits a degreewise finite projective resolution, the natural homothety map  $R \rightarrow \text{Hom}_R(C, C)$  is an isomorphism and  $\text{Ext}_R^{\geq 1}(C, C) = 0$ . Examples include the rank one free module and a dualizing (canonical) module when one exists. With this notion, the Auslander class and the Bass class with respect to a semidualizing  $R$ -module  $C$ , denoted by  $\mathcal{A}_C(R)$  and  $\mathcal{B}_C(R)$  respectively, can be defined and studied naturally. It is well known that there exists the following equivalence of categories:

$$\mathcal{A}_C(R) \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} \mathcal{B}_C(R)$$

Recently, to be a kind of generalization of the classes of Gorenstein projective and Gorenstein injective modules, denoted by  $\mathcal{G}(\mathcal{P})$  and  $\mathcal{G}(\mathcal{I})$  respectively, Geng and Ding [9] introduced the notions of the  $\mathcal{W}_P$ -Gorenstein and the  $\mathcal{W}_I$ -Gorenstein modules. Moreover, they obtained the following interesting equivalences of categories:

$$\mathcal{G}(\mathcal{P}) \cap \mathcal{A}_C(R) \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} \mathcal{G}(\mathcal{W}_P) \text{ and } \mathcal{G}(\mathcal{W}_I) \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} \mathcal{G}(\mathcal{I}) \cap \mathcal{B}_C(R)$$

where  $\mathcal{G}(\mathcal{W}_P)$  and  $\mathcal{G}(\mathcal{W}_I)$  denote the classes of  $\mathcal{W}_P$ -Gorenstein and  $\mathcal{W}_I$ -Gorenstein modules respectively. So it is naturally to ask if there exist some other classes satisfying the following diagram:

$$\mathcal{G}(\mathcal{F}) \cap \mathcal{A}_C(R) \begin{array}{c} \xrightarrow{C \otimes_R -} \\ \xleftarrow{\text{Hom}_R(C, -)} \end{array} ?$$

The motivation of the present article is the  $?$ .

We shall introduce, in section 3, the notion of the  $\mathcal{W}_F$ -Gorenstein modules, which plays the role of  $?$ . Combined with  $\mathcal{W}_P$ -Gorenstein and  $\mathcal{W}_I$ -Gorenstein modules, they can be treated from a similar aspect as the relationship among projective, injective and flat modules in classical homological algebra theory. An  $R$ -module  $M$  is said to be  $\mathcal{W}_F$ -Gorenstein if there exists an exact sequence

$$\mathbb{W}_\bullet = \cdots \rightarrow W_1 \rightarrow W_0 \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$$

in  $\mathcal{F}_C(R)$  such that  $M \cong \text{im}(W_0 \rightarrow W^0)$  and that  $\mathbb{W}_\bullet$  is  $\text{Hom}_R(\mathcal{P}_C(R), -)$  and  $\mathcal{I}_C(R) \otimes_R -$  exact, where  $\mathcal{F}_C(R)$ ,  $\mathcal{P}_C(R)$  and  $\mathcal{I}_C(R)$  denote the classes of  $C$ -flat,  $C$ -projective and  $C$ -injective modules respectively. Furthermore, we get the following theorem demonstrating the relationship between the classes  $\mathcal{G}(\mathcal{W}_F)$  and  $\mathcal{G}\mathcal{F}_C(R)$  (see Theorem 3.4):

**Theorem I** Let  $C$  be a semidualizing  $R$ -module. Then we have  $\mathcal{G}(\mathcal{W}_F) = \mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R)$ .

Also, the  $\mathcal{G}(\mathcal{W}_F)$ -projective dimension for any  $R$ -module will be investigated in this section.

In section 4, we first introduce the modules that arise from an iteration of the above construction. To wit, let  $\mathcal{G}^2(\mathcal{W}_F)$  denote the class of  $R$ -module  $M$  for which there exists an exact sequence

$$\mathbb{G}_\bullet = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots$$

in  $\mathcal{G}(\mathcal{W}_F)$  such that  $M \cong \text{im}(G_0 \rightarrow G^0)$  and that  $\mathbb{G}_\bullet$  is  $\text{Hom}_R(\mathcal{G}(\mathcal{W}_F), -)$  and  $\mathcal{G}(\mathcal{W}_F)^+ \otimes_R -$  exact. Similarly, we can also define  $R$ -modules which belong to  $\mathcal{G}^2(\mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R))$  or  $\mathcal{G}^2(\mathcal{F})$ . Although the definition defined above differs from the one appeared in [14], there is still a good correspondence. We then apply those techniques obtained in the former parts of this paper to get our results concerning the stability properties of Gorenstein categories (see Theorem 4.5, Corollary 4.6 and Corollary 4.7).

**Theorem II** Let  $R$  be a GF-closed ring and  $C$  be a semidualizing  $R$ -module. Then the following hold:

- (1)  $\mathcal{G}^2(\mathcal{W}_F) = \mathcal{G}(\mathcal{W}_F)$ .
- (2)  $\mathcal{G}^2(\mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R)) = \mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R)$ .
- (3)  $\mathcal{G}^2(\mathcal{F}) = \mathcal{G}(\mathcal{F})$ .

In the following part of this paper, let  $C$  be a semidualizing  $R$ -module and we mainly recall some necessary notions and definitions in the next section.

## 2. Notions and definitions

Let  $\mathcal{X} = \mathcal{X}(R)$ ,  $\mathcal{Y} = \mathcal{Y}(R)$  be classes of  $R$ -modules. We begin with the following definition.

**Definition 2.1.** We write  $\mathcal{X} \perp \mathcal{Y}$  if  $\text{Ext}_R^{\geq 1}(X, Y) = 0$  for each object  $X \in \mathcal{X}$  and each object  $Y \in \mathcal{Y}$ . Write  $\mathcal{X} \top \mathcal{Y}$  if  $\text{Tor}_{\geq 1}^R(X, Y) = 0$  for each object  $X \in \mathcal{X}$  and each object

$Y \in \mathcal{Y}$ . For an  $R$ -module  $M$ , write  $M \perp \mathcal{Y}$  if  $\text{Ext}_R^{\geq 1}(M, Y) = 0$  (resp.,  $\mathcal{Y} \perp M$  if  $\text{Ext}_R^{\geq 1}(Y, M) = 0$ ) for each object  $Y \in \mathcal{Y}$ . Write  $\mathcal{Y} \top M$  if  $\text{Tor}_{\geq 1}^R(Y, M) = 0$  for each object  $Y \in \mathcal{Y}$ . Following [15], we say that  $\mathcal{X}$  is a generator for  $\mathcal{Y}$  if  $\mathcal{X} \subseteq \mathcal{Y}$  and, for each object  $Y \in \mathcal{Y}$ , there exists a short exact sequence

$$0 \longrightarrow Y' \longrightarrow X \longrightarrow Y \longrightarrow 0$$

in  $\mathcal{Y}$  such that  $X \in \mathcal{X}$ . The class  $\mathcal{X}$  is a projective generator for  $\mathcal{Y}$  if  $\mathcal{X}$  is a generator for  $\mathcal{Y}$  and  $\mathcal{X} \perp \mathcal{Y}$ .

**Definition 2.2.** For any  $R$ -module  $M$ , we recall three types of resolutions.

- (1) [10, 1.5] A left  $\mathcal{X}$ -resolution of  $M$  is an exact sequence  $\mathbb{X} = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$  with  $X_n \in \mathcal{X}$  for all  $n \geq 0$ .
- (2) [10, 1.5] A right  $\mathcal{X}$ -resolution of  $M$  is an exact sequence  $\mathbb{X} = 0 \rightarrow M \rightarrow X^0 \rightarrow X^1 \rightarrow \cdots$  with  $X^n \in \mathcal{X}$  for all  $n \geq 0$ .

Now let  $\mathbb{X}$  be any (left or right)  $\mathcal{X}$ -resolution of  $M$ . We say that  $\mathbb{X}$  is co-proper if the sequence  $\text{Hom}_R(\mathbb{X}, Y)$  is exact for all  $Y \in \mathcal{X}$ .

(3) [16, 1.6] A degreewise finite projective (resp., free) resolution of  $M$  is a left projective (resp., free) resolution  $\mathbb{P}$  of  $M$  such that each  $P_i$  is finitely generated projective (resp., free). It is easy to verify that  $M$  admits a degreewise finite projective resolution if and only if  $M$  admits a degreewise finite free resolution.

**Definition 2.3.** The  $\mathcal{X}$ -projective dimension of an  $R$ -module  $M$  is defined as

$$\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \mid X_n \neq 0\} \mid \mathbb{X} \text{ is a left } \mathcal{X}\text{-resolution of } M\}.$$

Dually, we can also define the  $\mathcal{X}$ -injective dimension of  $M$ .

The next lemma has a standard proof.

**Lemma 2.4.** Let  $M$  be an  $R$ -module. Consider the following exact sequence in  $\mathcal{X}$ :

$$\mathbb{X} = \cdots \longrightarrow X_1 \xrightarrow{\delta_1^{\mathbb{X}}} X_0 \xrightarrow{\delta_0^{\mathbb{X}}} X_{-1} \longrightarrow \cdots$$

Then we have the following hold:

- (1) Assume  $M \perp \mathcal{X}$ . If  $\mathbb{X}$  is  $\text{Hom}_R(M, -)$  exact, then  $\text{Ext}_R^{\geq 1}(M, \text{im}(\delta_i^{\mathbb{X}})) = 0$  for all  $i$ . Conversely, if  $\text{Ext}_R^1(M, \text{im}(\delta_i^{\mathbb{X}})) = 0$  for all  $i$ , then  $\mathbb{X}$  is  $\text{Hom}_R(M, -)$  exact.
- (2) Assume  $M \top \mathcal{X}$ . If  $\mathbb{X}$  is  $M \otimes_R -$  exact, then  $\text{Tor}_{\geq 1}^R(M, \text{im}(\delta_i^{\mathbb{X}})) = 0$  for all  $i$ . Conversely, if  $\text{Tor}_1^R(M, \text{im}(\delta_i^{\mathbb{X}})) = 0$  for all  $i$ , then  $\mathbb{X}$  is  $M \otimes_R -$  exact.

**Definition 2.5.** An  $R$ -module  $M$  is said to be *Gorenstein flat* if there exists an exact sequence

$$\mathbb{X} = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$$

in  $R\text{-Mod}$  with each  $F_i$  and  $F^i$  flat such that  $M \cong \text{im}(F_0 \rightarrow F^0)$  and that  $\mathbb{X}$  is  $I \otimes_R -$  exact for any injective  $R$ -module  $I$ . The exact sequence  $\mathbb{X}$  is called *complete flat resolution* of  $M$ .

In the following, we denote the class of Gorenstein flat modules by  $\mathcal{G}(\mathcal{F})$ .

**Definition 2.6.** Let  $R$  be a ring. We call  $R$  *GF-closed* if the class of Gorenstein flat  $R$ -modules is closed under extensions, that is, if  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  is an exact sequence with  $X$  and  $Z$  Gorenstein flat modules, then  $Y$  is also Gorenstein flat.

It follows from [2] that the class of GF-closed rings includes strictly the one of coherent rings and the one of rings of finite weak global dimension.

**Definition 2.7.** An  $R$ -module is  *$C$ -projective* (resp.,  *$C$ -flat*) if it has the form  $C \otimes_R P$  for some projective (resp., flat)  $R$ -module  $P$ . An  $R$ -module is  *$C$ -injective* if it has the form  $\text{Hom}_R(C, I)$  for some injective  $R$ -module  $I$ . We set

$$\begin{aligned} \mathcal{P}_C(R) &= \{C \otimes_R P \mid P \text{ is a projective } R\text{-module}\} \\ \mathcal{F}_C(R) &= \{C \otimes_R F \mid F \text{ is a flat } R\text{-module}\} \\ \mathcal{I}_C(R) &= \{\text{Hom}_R(C, I) \mid I \text{ is an injective } R\text{-module}\}. \end{aligned}$$

**Remark 2.8.** The classes defined above are studied extensively in [12]. From there we know that

- (1) The classes  $\mathcal{F}_C(R)$  and  $\mathcal{P}_C(R)$  are closed under arbitrary direct sums and summands and if  $R$  is coherent, then  $\mathcal{F}_C(R)$  is also closed under arbitrary direct products.
- (2) The class  $\mathcal{I}_C(R)$  is closed under arbitrary direct products and summands.

**Definition 2.9.** An  $R$ -module  $N$  is said to be  *$G_C$ -injective* ( $G_C\text{-inj}$  for short) if there exists an exact sequence

$$\mathbb{Y} = \cdots \rightarrow \text{Hom}_R(C, I^1) \rightarrow \text{Hom}_R(C, I^0) \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots$$

in  $R\text{-Mod}$  with each  $I_i$  and  $I^i$  injective such that  $N \cong \text{im}(\text{Hom}_R(C, I^0) \rightarrow I_0)$  and that  $\mathbb{Y}$  is  $\text{Hom}_R(\mathcal{I}_C(R), -)$  exact. The exact sequence  $\mathbb{Y}$  is called *complete  $\mathcal{I}_C\mathcal{I}$ -resolution* of  $N$ .

An  $R$ -module  $T$  is said to be  *$G_C$ -flat* if there exists an exact sequence

$$\mathbb{Z} = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots$$

in  $R\text{-Mod}$  with each  $F_i$  and  $F^i$  flat such that  $M \cong \text{im}(F_0 \rightarrow C \otimes_R F^0)$  and that  $\mathbb{Z}$  is  $\mathcal{I}_C(R) \otimes_R -$  exact. The exact sequence  $\mathbb{Z}$  is called complete  $\mathcal{FF}_C$ -resolution of  $T$ .

We will denote the classes of  $G_C\text{-inj}$  and  $G_C\text{-flat}$  modules by  $\mathcal{GI}_C(R)$  and  $\mathcal{GF}_C(R)$  respectively.

**Remark 2.10.** Similar to the proofs in [16] we can easily get that

- (1) Every  $C$ -injective  $R$ -module is  $G_C\text{-inj}$  and the class  $\mathcal{GI}_C(R)$  is coresolving and closed under arbitrary direct products and summands.
- (2) Every  $C$ -flat  $R$ -module is  $G_C\text{-flat}$  and the class  $\mathcal{GF}_C(R)$  is closed under arbitrary direct sums.
- (3) Every kernel and cokernel of a complete  $\mathcal{I}_C\mathcal{I}$ -resolution (resp.,  $\mathcal{FF}_C$ -resolution) belongs to  $\mathcal{GI}_C(R)$  (resp.,  $\mathcal{GF}_C(R)$ ).

By using the definition of  $G_C\text{-flat}$  modules and Remark 2.10, the proof of the next lemma is a standard argument.

**Lemma 2.11.** The following are equivalent for an  $R$ -module  $M$ :

- (1)  $M$  is  $G_C\text{-flat}$ .
- (2)  $M$  satisfies the following two conditions:
  - (i)  $\mathcal{I}_C(R) \perp M$  and
  - (ii) There exists an exact sequence  $0 \rightarrow M \rightarrow C \otimes_R F^0 \rightarrow C \otimes_R F^1 \rightarrow \cdots$  in  $R\text{-Mod}$  with each  $F^i$  flat such that  $\mathcal{I}_C(R) \otimes_R -$  leaves it exact.
- (3) There exists a short exact sequence  $0 \rightarrow M \rightarrow C \otimes_R F \rightarrow G \rightarrow 0$  in  $R\text{-Mod}$  with  $F$  flat and  $G \in \mathcal{GF}_C(R)$ .

**Definition 2.12.** The Auslander class  $\mathcal{A}_C(R)$  with respect to  $C$  consists of all  $R$ -modules  $M$  satisfying

- (1)  $\text{Tor}_{\geq 1}^R(C, M) = 0 = \text{Ext}_R^{\geq 1}(C, C \otimes_R M)$  and
- (2) The natural evaluation map  $\mu_{CCM} : M \rightarrow \text{Hom}_R(C, C \otimes_R M)$  is an isomorphism.

Dually, the Bass class  $\mathcal{B}_C(R)$  with respect to  $C$  consists of all  $R$ -modules  $N$  satisfying

- (1)  $\text{Ext}_R^{\geq 1}(C, N) = 0 = \text{Tor}_{\geq 1}^R(C, \text{Hom}_R(C, N))$  and
- (2) The natural evaluation map  $\nu_{CCN} : C \otimes_R \text{Hom}_R(C, N) \rightarrow N$  is an isomorphism.

We now state some basic results about the classes  $\mathcal{A}_C(R)$  and  $\mathcal{B}_C(R)$ .

**Lemma 2.13** (12). *The following hold:*

- (1) *If any two  $R$ -modules in a short exact sequence are in  $\mathcal{A}_C(R)$ , respectively  $\mathcal{B}_C(R)$ , then so is the third.*
- (2) *The class  $\mathcal{A}_C(R)$  contains all modules of finite flat dimension and those of finite  $\mathcal{I}_C$ -injective dimension. The class  $\mathcal{B}_C(R)$  contains all modules of finite injective dimension and those of finite  $\mathcal{F}_C$ -projective dimension.*

To be a direct corollary of [12, Theorem 6.4], we have the following lemma:

**Lemma 2.14.**  $\mathcal{P}_C(R) \perp \mathcal{B}_C(R)$ ,  $\mathcal{A}_C(R) \perp \mathcal{I}_C(R)$  and  $\mathcal{A}_C(R) \top \mathcal{F}_C(R)$ .

### 3. $\mathcal{W}_F$ -Gorenstein modules

Now we introduce and investigate the notion of  $\mathcal{W}_F$ -Gorenstein modules and the corresponding projective dimension for any  $R$ -Module.

**Definition 3.1.** *An  $R$ -module  $M$  is said to be  $\mathcal{W}_F$ -Gorenstein if there exists an exact sequence*

$$\mathbb{W}_\bullet = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

*in  $\mathcal{F}_C(R)$  such that  $M \cong \text{im}(W_0 \rightarrow W^0)$  and that  $\mathbb{W}_\bullet$  is  $\text{Hom}_R(\mathcal{P}_C(R), -)$  and  $\mathcal{I}_C(R) \otimes_R -$  exact.*

It is clear that each module in  $\mathcal{F}_C(R)$  is  $\mathcal{W}_F$ -Gorenstein, and every kernel and cokernel of  $\mathbb{W}_\bullet$  is  $\mathcal{W}_F$ -Gorenstein.

In the following, we denote by  $\mathcal{G}(\mathcal{W}_F)$  the class of  $\mathcal{W}_F$ -Gorenstein modules.

**Proposition 3.2.**  $\mathcal{P}_C(R) \perp \mathcal{G}(\mathcal{W}_F)$  and  $\mathcal{I}_C(R) \top \mathcal{G}(\mathcal{W}_F)$ .

*Proof.* It follows directly from Lemma 2.4 and Lemma 2.14.  $\square$

**Proposition 3.3.** *Let  $\mathbb{W}_\bullet = \cdots \rightarrow W_1 \rightarrow W_0 \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$  be an exact sequence in  $\mathcal{F}_C(R)$  and  $M \cong \text{im}(W_0 \rightarrow W^0)$ . Then  $\mathbb{W}_\bullet$  is  $\text{Hom}_R(\mathcal{P}_C(R), -)$  exact if and only if  $M \in \mathcal{B}_C(R)$ .*

*Proof.* Suppose that  $M \in \mathcal{B}_C(R)$ . By Lemma 2.13, every kernel and cokernel of  $\mathbb{W}_\bullet$  is in  $\mathcal{B}_C(R)$ , and so  $\mathbb{W}_\bullet$  is  $\text{Hom}_R(\mathcal{P}_C(R), -)$  exact by Lemma 2.4 and Lemma 2.14.

Conversely, if  $\mathbb{W}_\bullet$  is  $\text{Hom}_R(\mathcal{P}_C(R), -)$  exact, then by Lemma 2.4 and Lemma 2.14, we have  $\mathcal{P}_C(R) \perp M$ . Thus, there exists an exact sequence

$$\cdots \rightarrow W_1 \rightarrow W_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

in  $R\text{-Mod}$  with each  $I^i$  injective such that  $M \cong \text{im}(W_0 \rightarrow I^0)$  and that  $\text{Hom}_R(\mathcal{P}_C(R), -)$  leaves it exact. Hence  $M \in \mathcal{B}_C(R)$  by [12, Theorem 6.1].  $\square$

Now we are in position to prove the result of linking the classes  $\mathcal{GF}_C(R)$  and  $\mathcal{G}(\mathcal{W}_F)$  together.

**Theorem 3.4.** *Let  $M$  be an  $R$ -module. Then  $M \in \mathcal{G}(\mathcal{W}_F)$  if and only if  $M \in \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$ .*

*Proof.*  $(\Rightarrow)$  Let  $M \in \mathcal{G}(\mathcal{W}_F)$ . We first have  $\mathcal{I}_C(R) \top M$  by Proposition 3.2. Then  $M \in \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$  by Lemma 2.11 and Proposition 3.3.

$(\Leftarrow)$  Let  $M \in \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$ . Since  $M \in \mathcal{GF}_C(R)$ , we have that  $\mathcal{I}_C(R) \top M$  and there exists an exact sequence

$$0 \rightarrow M \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$$

in  $R\text{-Mod}$  with each  $W^i \in \mathcal{F}_C(R)$  such that  $\mathcal{I}_C(R) \otimes_R -$  leaves it exact. On the other hand, since  $M \in \mathcal{B}_C(R)$ , it is easy to verify that  $M$  has a proper left  $\mathcal{P}_C(R)$ -resolution

$$\cdots \rightarrow V_1 \rightarrow V_0 \rightarrow M \rightarrow 0.$$

It follows from Lemma 2.13 and Lemma 2.14 that  $\mathcal{I}_C(R) \otimes_R -$  leaves it exact. Thus, we have the following exact sequence:

$$\cdots \rightarrow V_1 \rightarrow V_0 \rightarrow W^0 \rightarrow W^1 \rightarrow \cdots$$

such that  $M \cong \text{im}(V_0 \rightarrow W^0)$ . By Proposition 3.3, we know that  $\text{Hom}_R(\mathcal{P}_C(R), -)$  leaves it exact. It follows that  $M \in \mathcal{G}(\mathcal{W}_F)$ .  $\square$

**Theorem 3.5.** *There exists equivalence of categories:*

$$\mathcal{G}(\mathcal{F}) \cap \mathcal{A}_C(R) \begin{array}{c} \xleftarrow{\quad C \otimes_R - \quad} \\[-1ex] \xrightarrow{\quad \text{Hom}_R(C, -) \quad} \end{array} \mathcal{G}(\mathcal{W}_F).$$

*Proof.* We first show that the functor  $\text{Hom}_R(C, -)$  maps  $\mathcal{G}(\mathcal{W}_F)$  to  $\mathcal{G}(\mathcal{F}) \cap \mathcal{A}_C(R)$ . Assume that  $M \in \mathcal{G}(\mathcal{W}_F)$ . Then there exists an exact sequence

$$\mathbb{W}_\bullet = \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow W^0 \longrightarrow W^1 \longrightarrow \cdots$$

in  $\mathcal{F}_C(R)$  such that  $M \cong \text{im}(W_0 \rightarrow W^0)$  and that  $\mathbb{W}_\bullet$  is  $\text{Hom}_R(\mathcal{P}_C(R), -)$  and  $\mathcal{I}_C(R) \otimes_R -$  exact. So  $M \in \mathcal{B}_C(R)$  by Theorem 3.4, and hence every kernel and cokernel of  $\mathbb{W}_\bullet$  is in  $\mathcal{B}_C(R)$  by Lemma 2.13. Thus,  $\text{Hom}_R(C, \mathbb{W}_\bullet)$  is exact, moreover,  $\text{Hom}_R(C, M) \in \mathcal{A}_C(R)$  by [12, Proposition 4.1]. On the other hand, suppose that  $W_i \cong C \otimes_R F_i$  and  $W^i \cong C \otimes_R F^i$ , where each  $F_i$  and  $F^i$  flat. Then we have the following exact sequence in  $R\text{-Mod}$ :

$$\text{Hom}_R(C, \mathbb{W}_\bullet) = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

such that  $\text{Hom}_R(C, M) \cong \text{im}(F_0 \rightarrow F^0)$ . For each injective  $R$ -module  $I$ , we have

$$I \otimes_R \text{Hom}_R(C, \mathbb{W}_\bullet) \cong C \otimes_R \text{Hom}_R(C, I) \otimes_R \text{Hom}_R(C, \mathbb{W}_\bullet) \cong \text{Hom}_R(C, I) \otimes_R \mathbb{W}_\bullet$$

is exact. Hence,  $\text{Hom}_R(C, M) \in \mathcal{G}(\mathcal{F})$ .

The proof of  $C \otimes_R -$  maps  $\mathcal{G}(\mathcal{F}) \cap \mathcal{A}_C(R)$  to  $\mathcal{G}(\mathcal{W}_F)$  is similar.  $\square$

**Corollary 3.6.** *Let  $R$  be a GF-closed ring. Then the class  $\mathcal{G}(\mathcal{W}_F)$  is closed under extensions, kernels of epimorphisms and direct summands.*

*Proof.* We first show that the class  $\mathcal{G}(\mathcal{W}_F)$  is closed under extensions when  $R$  is GF-closed. Consider the following short exact sequence:

$$0 \longrightarrow M \longrightarrow N \longrightarrow K \longrightarrow 0$$

with  $M$  and  $N$  belong to  $\mathcal{G}(\mathcal{W}_F)$ . Since  $M \in \mathcal{B}_C(R)$  by Theorem 3.4, we get the next exact sequence

$$0 \longrightarrow \text{Hom}_R(C, M) \longrightarrow \text{Hom}_R(C, N) \longrightarrow \text{Hom}_R(C, K) \longrightarrow 0.$$

It follows from Theorem 3.5 that  $\text{Hom}_R(C, M)$  and  $\text{Hom}_R(C, K)$  belong to  $\mathcal{G}(\mathcal{F}) \cap \mathcal{A}_C(R)$ . Thus,  $\text{Hom}_R(C, N)$  belongs to  $\mathcal{G}(\mathcal{F}) \cap \mathcal{A}_C(R)$ . On the other hand, since  $N \in \mathcal{B}_C(R)$  by Lemma 2.13 and Theorem 3.4,  $N \cong C \otimes_R \text{Hom}_R(C, N) \in \mathcal{G}(\mathcal{W}_F)$  by Theorem 3.5, as desired.

The proof of the class  $\mathcal{G}(\mathcal{W}_F)$  is closed under kernels of epimorphisms and direct summands is similar to [2, Theorem 2.3 and Corollary 2.6].  $\square$

**Lemma 3.7.** *Let  $R$  be a GF-closed ring. For every short exact sequence  $0 \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$  in  $R\text{-Mod}$  with  $G_0, G_1 \in \mathcal{G}(\mathcal{W}_F)$ , if  $\text{Tor}_1^R(\mathcal{I}_C(R), M) = 0$ , then  $M \in \mathcal{G}(\mathcal{W}_F)$ .*

*Proof.* By the fact that the class  $\mathcal{F}_C(R)$  is closed direct summands and [14, Lemma 4.1], the proof of the lemma is similar to [2, Theorem 2.3].  $\square$

**Theorem 3.8.** *Let  $R$  be a GF-closed ring and  $M$  an  $R$ -module with finite  $\mathcal{G}(\mathcal{W}_F)$ -projective dimension and let  $n$  be a non-negative integer. Then the following are equivalent:*

- (1)  $\mathcal{G}(\mathcal{W}_F)\text{-pd}_R(M) \leq n$ .
- (2) For every non-negative integer  $t$  such that  $0 \leq t \leq n$ , there exists an exact sequence  $0 \rightarrow W_n \rightarrow \cdots \rightarrow W_1 \rightarrow W_0 \rightarrow M \rightarrow 0$  in  $R\text{-Mod}$  such that  $W_t \in \mathcal{G}(\mathcal{W}_F)$  and  $W_i \in \mathcal{F}_C(R)$  for  $i \neq t$ .
- (3) There exists a short exact sequence  $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$  in  $R\text{-Mod}$  with  $G \in \mathcal{G}(\mathcal{W}_F)$  and  $\mathcal{F}_C(R)\text{-pd}_R(K) \leq n-1$ .
- (4) There exists a short exact sequence  $0 \rightarrow M \rightarrow K' \rightarrow G' \rightarrow 0$  in  $R\text{-Mod}$  with  $G' \in \mathcal{G}(\mathcal{W}_F)$  and  $\mathcal{F}_C(R)\text{-pd}_R(K') \leq n$ .
- (5) There exists an exact sequence  $0 \rightarrow G \rightarrow V_{n-1} \rightarrow \cdots \rightarrow V_0 \rightarrow M \rightarrow 0$  in  $R\text{-Mod}$  with  $G \in \mathcal{G}(\mathcal{W}_F)$  and  $V_i \in \mathcal{P}_C(R)$  for all  $0 \leq i \leq n-1$ .
- (6) For every exact sequence  $0 \rightarrow K_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0$  in  $R\text{-Mod}$  with  $G_i \in \mathcal{G}(\mathcal{W}_F)$  for all  $0 \leq i \leq n-1$ , then also  $K_n \in \mathcal{G}(\mathcal{W}_F)$ .
- (7)  $\text{Tor}_R^{n+j}(U, M) = 0$  for all  $j \geq 1$  and all  $U \in \mathcal{I}_C(R)$ .
- (8)  $\text{Tor}_R^{n+j}(U, M) = 0$  for all  $j \geq 1$  and all  $U$  with  $\mathcal{I}_C(R)\text{-id}_R(U) < \infty$ .

Furthermore, we have that

$$\begin{aligned} \mathcal{G}(\mathcal{W}_F)\text{-pd}_R(M) &= \sup\{ n \in \mathbb{N} \mid \text{Tor}_R^n(U, M) \neq 0 \text{ for some } U \in \mathcal{I}_C(R) \} \\ &= \sup\{ n \in \mathbb{N} \mid \text{Tor}_R^n(U, M) \neq 0 \text{ for some } U \text{ with } \mathcal{I}_C(R)\text{-id}_R(U) < \infty \}. \end{aligned}$$

*Proof.* (2)  $\Rightarrow$  (3)  $\Rightarrow$  (1) and (6)  $\Rightarrow$  (1) are clear.

(1)  $\Rightarrow$  (7)  $\Rightarrow$  (8) follow from usual dimension shifting argument.

(1)  $\Rightarrow$  (2) Since the class  $\mathcal{G}(\mathcal{W}_F)$  is closed under extensions by Corollary 3.6, the proof is similar to [13, Theorem 2.6].

(3)  $\Rightarrow$  (4) Since  $G \in \mathcal{G}(\mathcal{W}_F)$ , there exist a short exact sequence  $0 \rightarrow G \rightarrow W \rightarrow G' \rightarrow 0$  in  $R\text{-Mod}$  with  $W \in \mathcal{F}_C(R)$  and  $G' \in \mathcal{G}(\mathcal{W}_F)$ . Now consider the following push-out diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & M \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & K & \longrightarrow & W & \longrightarrow & K' \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & G' & \equiv & G' & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

From the second row in the above diagram, we know  $\mathcal{F}_C(R)\text{-pd}_R(K') \leq n$ . So the third column is as desired.

(4)  $\Rightarrow$  (3) Since  $\mathcal{F}_C(R)\text{-pd}_R(K') \leq n$ , there exist a short exact sequence  $0 \rightarrow K \rightarrow W \rightarrow K' \rightarrow 0$  in  $R\text{-Mod}$  with  $W \in \mathcal{F}_C(R)$  and  $\mathcal{F}_C(R)\text{-pd}_R(K) \leq n-1$ . Then consider the following pullback diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & K & \equiv & K & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G & \longrightarrow & W & \longrightarrow & G' \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & M & \longrightarrow & K' & \longrightarrow & G' \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

From the second row, we know that  $G \in \mathcal{G}(\mathcal{W}_F)$  by Corollary 3.6. So the first column is as desired.

(1)  $\Rightarrow$  (5) It suffices to prove the case for  $n = 1$ . Assume that  $\mathcal{G}(\mathcal{W}_F)\text{-pd}_R(M) \leq 1$ . Then there exists a short exact sequence  $0 \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$  in  $R\text{-Mod}$  with  $G_0, G_1 \in \mathcal{G}(\mathcal{W}_F)$ . By Theorem 3.4, we know that  $G_0 \in \mathcal{B}_C(R)$ . Thus, it is easy to verify that there exists a short exact sequence  $0 \rightarrow G'_0 \rightarrow V \rightarrow G_0 \rightarrow 0$  in  $R\text{-Mod}$  such that

$V \in \mathcal{P}_C(R)$ , then also  $V \in \mathcal{G}(\mathcal{W}_F)$ . By Corollary 3.6, we have  $G'_0 \in \mathcal{G}(\mathcal{W}_F)$ . Now consider the following pullback diagram:

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
& G'_0 & \xlongequal{\quad} & G'_0 & & & \\
& \downarrow & & \downarrow & & & \\
0 & \longrightarrow & G & \longrightarrow & V & \longrightarrow & M \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & M \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& 0 & & 0 & & &
\end{array}$$

From the first column in the above diagram, we know that  $G \in \mathcal{G}(\mathcal{W}_F)$  by Corollary 3.6. So the middle row is as desired.

(5)  $\Rightarrow$  (6) Let  $0 \rightarrow K_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0$  be an exact sequence in  $R\text{-Mod}$  with each  $G_i \in \mathcal{G}(\mathcal{W}_F)$ , then also  $G_i \in \mathcal{B}_C(R)$  by Theorem 3.4. Thus,  $K_n \in \mathcal{B}_C(R)$  and  $\mathcal{P}_C(R) \perp K_n$  by Lemma 2.13 and Lemma 2.14. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & G_n & \longrightarrow & V_{n-1} & \longrightarrow & \cdots & \longrightarrow & V_1 & \longrightarrow & V_0 & \longrightarrow & M & \longrightarrow 0 \\
& & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \parallel & \\
0 & \longrightarrow & K_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow 0
\end{array}$$

Thus, the mapping cone

$$0 \longrightarrow G_n \longrightarrow V_{n-1} \oplus K_n \longrightarrow \cdots \longrightarrow V_0 \oplus G_1 \longrightarrow G_0 \longrightarrow 0$$

is exact. It follows from Corollary 3.6 that  $K_n \in \mathcal{G}(\mathcal{W}_F)$ .

(8)  $\Rightarrow$  (1) By Lemma 3.7, the proof is similar to [2, Theorem 2.8].

The last claim is an immediate consequence of the equivalent of (1), (7) and (8).  $\square$

Let  $n$  be a non-negative integer. In what follows, we denote by  $\mathcal{G}\text{-flat}_{\leq n}$  (resp.,  $\mathcal{G}_C\text{-flat}_{\leq n}$ ) the class of modules with finite Gorenstein flat (resp.,  $\mathcal{G}(\mathcal{W}_F)$ -projective) dimension at most  $n$ .

**Theorem 3.9.** (Foxby equivalence) Let  $\mathcal{F}(R)$  be the class of flat modules. There are equivalences of categories:

$$\begin{array}{ccc}
\mathcal{F}(R) & \xrightleftharpoons[C \otimes_R -]{\text{Hom}_R(C, -)} & \mathcal{F}_C(R) \\
\downarrow & & \downarrow \\
\mathcal{G}(\mathcal{F}) \cap \mathcal{A}_C(R) & \xrightleftharpoons[C \otimes_R -]{\text{Hom}_R(C, -)} & \mathcal{G}(\mathcal{W}_F) \\
\downarrow & & \downarrow \\
\mathcal{G}\text{-flat}_{\leq n} \cap \mathcal{A}_C(R) & \xrightleftharpoons[C \otimes_R -]{\text{Hom}_R(C, -)} & \mathcal{G}_C\text{-flat}_{\leq n} \\
\downarrow & & \downarrow \\
\mathcal{A}_C(R) & \xrightleftharpoons[C \otimes_R -]{\text{Hom}_R(C, -)} & \mathcal{B}_C(R)
\end{array}$$

*Proof.* Let  $M$  be an  $R$ -module. It suffices to prove the equivalence of categories of the third rows in the above diagram.

For the third row, it suffices to prove the case for  $n = 1$ . Assume that  $M \in \mathcal{G}_C\text{-flat}_{\leq 1}$ . Then there exists a short exact sequence

$$0 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

in  $R\text{-Mod}$  with  $G_0, G_1 \in \mathcal{G}(\mathcal{W}_F)$ . Since  $G_1 \in \mathcal{B}_C(R)$  by Theorem 3.4, we have the following exact sequence in  $R\text{-Mod}$ :

$$0 \longrightarrow \text{Hom}_R(C, G_1) \longrightarrow \text{Hom}_R(C, G_0) \longrightarrow \text{Hom}_R(C, M) \longrightarrow 0$$

with  $\text{Hom}_R(C, G_0), \text{Hom}_R(C, G_1) \in \mathcal{G}(\mathcal{F}) \cap \mathcal{A}_C(R)$  by Theorem 3.5. Hence, by Lemma 2.13,  $\text{Hom}_R(C, M) \in \mathcal{G}\text{-flat}_{\leq 1} \cap \mathcal{A}_C(R)$ .

Conversely, assume that  $M \in \mathcal{G}\text{-flat}_{\leq 1} \cap \mathcal{A}_C(R)$ . Then there exists a short exact sequence  $0 \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$  in  $R\text{-Mod}$  with  $G_0, G_1 \in \mathcal{G}(\mathcal{F}) \cap \mathcal{A}_C(R)$ . Since  $M \in \mathcal{A}_C(R)$  by Lemma 2.13,  $\text{Tor}_R^{\geq 1}(C, M) = 0$ . Thus, there exists a short exact sequence

$$0 \longrightarrow C \otimes_R G_1 \longrightarrow C \otimes_R G_0 \longrightarrow C \otimes_R M \longrightarrow 0$$

in  $R\text{-Mod}$ . By Theorem 3.5, we know that  $C \otimes_R G_0, C \otimes_R G_1 \in \mathcal{G}(\mathcal{W}_F)$ . Hence,  $C \otimes_R M \in \mathcal{G}_C\text{-flat}_{\leq 1}$ .  $\square$

#### 4. Stability of Categories

We start with the following definition.

**Definition 4.1.** Let  $M$  be an  $R$ -module and  $n \geq 2$  a integer. We say that  $M \in \mathcal{G}^n(\mathcal{W}_F)$  if there exists an exact sequence

$$\mathbb{G}_\bullet = \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow G^0 \longrightarrow G^1 \longrightarrow \cdots$$

in  $\mathcal{G}^{n-1}(\mathcal{W}_F)$  such that  $M \cong \text{im}(G_0 \rightarrow G^0)$  and that  $\mathbb{G}_\bullet$  is  $\text{Hom}_R(\mathcal{G}^{n-1}(\mathcal{W}_F), -)$  and  $\mathcal{G}^{n-1}(\mathcal{W}_F)^+ \otimes_R -$  exact.

Set  $\mathcal{G}^0(\mathcal{W}_F) = \mathcal{F}_C(R)$ ,  $\mathcal{G}^1(\mathcal{W}_F) = \mathcal{G}(\mathcal{W}_F)$ . One can easily check that there is a contain  $\mathcal{G}^n(\mathcal{W}_F) \subseteq \mathcal{G}^{n+1}(\mathcal{W}_F)$  for all  $n \geq 0$ .

Similarly, we can also define modules which belong to  $\mathcal{G}^n(\mathcal{G}\mathcal{F}_C(R) \cap \mathcal{B}_C(R))$  or  $\mathcal{G}^n(\mathcal{F})$  for  $n \geq 2$ .

**Lemma 4.2.**  $\mathcal{P}_C(R) \perp \mathcal{G}^2(\mathcal{W}_F)$  and  $\mathcal{I}_C(R) \top \mathcal{G}^2(\mathcal{W}_F)$ .

*Proof.* It follows directly from Lemma 2.4, Proposition 3.2 and the fact that  $\mathcal{P}_C(R) \subseteq \mathcal{G}(\mathcal{W}_F)$  and  $\mathcal{I}_C(R) \subseteq \mathcal{G}(\mathcal{W}_F)^+$ .  $\square$

**Lemma 4.3.** Let  $R$  be a GF-closed ring. Then  $\mathcal{P}_C(R)$  is a projective generator for  $\mathcal{G}(\mathcal{W}_F)$ .

*Proof.* Let  $M$  be an  $R$ -module and  $M \in \mathcal{G}(\mathcal{W}_F)$ . So  $M \in \mathcal{B}_C(R)$  by Theorem 3.4. Thus, we have a short exact sequence

$$0 \longrightarrow M' \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$$

in  $R\text{-Mod}$  with  $P$  projective. By Corollary 3.6, we know that  $M' \in \mathcal{G}(\mathcal{W}_F)$ . On the other hand, it follows from Proposition 3.2 that  $\mathcal{P}_C(R) \perp \mathcal{G}(\mathcal{W}_F)$ . Hence,  $\mathcal{P}_C(R)$  is a projective generator for  $\mathcal{G}(\mathcal{W}_F)$ .  $\square$

**Lemma 4.4.** Let  $R$  be a GF-closed ring and let  $M$  be an  $R$ -module which belongs to  $\mathcal{G}^2(\mathcal{W}_F)$ . Then  $M$  admits a proper left  $\mathcal{P}_C(R)$ -resolution.

*Proof.* It follows directly from the definition of modules which belong to  $\mathcal{G}^2(\mathcal{W}_F)$ , Lemma 4.2, Lemma 4.3 and [15, Lemma 2.2(b)].  $\square$

**Theorem 4.5.** Let  $R$  be a GF-closed ring. We have  $\mathcal{G}^n(\mathcal{W}_F) = \mathcal{G}(\mathcal{W}_F)$  for all  $n \geq 1$ .

*Proof.* It suffices to prove the case for  $n = 2$ . Let  $M$  be an  $R$ -module and  $M \in \mathcal{G}^2(\mathcal{W}_F)$ . Following from Lemma 4.4, we have the following exact sequence in  $R\text{-Mod}$ :

$$(\alpha) = \cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow M \longrightarrow 0$$

with each  $P_i$  projective such that  $\text{Hom}_R(\mathcal{P}_C(R), -)$  leaves it exact. By Lemma 2.4, Lemma 2.14 and Lemma 4.2, we get that  $\mathcal{I}_C(R) \otimes_R -$  leaves  $(\alpha)$  exact as well.

On the other hand, since  $M \in \mathcal{G}^2(\mathcal{W}_F)$ , there exists a short exact sequence  $0 \rightarrow M \rightarrow G \rightarrow M' \rightarrow 0$  in  $R\text{-Mod}$  with  $G \in \mathcal{G}(\mathcal{W}_F)$  and  $M' \in \mathcal{G}^2(\mathcal{W}_F)$ . Since  $G \in \mathcal{G}(\mathcal{W}_F)$ , there exists a short exact sequence  $0 \rightarrow G \rightarrow C \otimes_R F^0 \rightarrow G' \rightarrow 0$  in  $R\text{-Mod}$  with  $F^0$  flat and  $G' \in \mathcal{G}(\mathcal{W}_F)$ . Then we have the following push-out diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & M' \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & M & \longrightarrow & C \otimes_R F^0 & \longrightarrow & K \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & G' & \xlongequal{\quad} & G' & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

Consider the following short exact sequence coming from the middle row of the above diagram:

$$(\beta) = 0 \longrightarrow M \longrightarrow C \otimes_R F^0 \longrightarrow K \longrightarrow 0$$

From the third column of the above push-out diagram, we know that  $\mathcal{I}_C(R) \vdash K$  by Proposition 3.2 and Lemma 4.2. Thus,  $(\beta)$  is  $\text{Hom}_R(\mathcal{P}_C(R), -)$  and  $\mathcal{I}_C(R) \otimes_R -$  exact. If we now can construct a short exact sequence

$$(\eta) = 0 \longrightarrow K \longrightarrow C \otimes_R F^1 \longrightarrow K' \longrightarrow 0$$

in  $R\text{-Mod}$  with  $F^1$  flat and  $K'$  a module with the same property as  $K$  (that is, there exists a short exact sequence  $(\mu) = 0 \rightarrow M'' \rightarrow K' \rightarrow H'' \rightarrow 0$  in  $R\text{-Mod}$  with  $M'' \in \mathcal{G}^2(\mathcal{W}_F)$  and  $H'' \in \mathcal{G}(\mathcal{W}_F)$ ), then the following exact sequence can be constructed recursively:

$$\begin{array}{ccccccc}
(\gamma) = & 0 & \longrightarrow & K & \longrightarrow & C \otimes_R F^1 & \longrightarrow C \otimes_R F^2 \longrightarrow \cdots \\
& & & & & \searrow \swarrow & \\
& & & & K' & & \\
& & & & \searrow \swarrow & & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{array}$$

From the middle row of the above push-out diagram and  $(\mu)$ , we get  $\mathcal{P}_C(R) \perp K$  and  $\mathcal{I}_C(R) \top K'$  by Proposition 3.2, Lemma 2.14 and Lemma 4.2. Then we have that  $(\eta)$  is  $\text{Hom}_R(\mathcal{P}_C(R), -)$  and  $\mathcal{I}_C(R) \otimes_R -$  exact. So is  $(\gamma)$ . Assembling the sequence  $(\alpha)$ ,  $(\beta)$  and  $(\gamma)$ , we get the following exact sequence in  $R\text{-Mod}$ :

$$\cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots$$

such that  $M \cong \text{im}(C \otimes_R P_0 \rightarrow C \otimes_R F^0)$  and that  $\text{Hom}_R(\mathcal{P}_C(R), -)$  and  $\mathcal{I}_C(R) \otimes_R -$  leave it exact. It follows that  $M \in \mathcal{G}(\mathcal{W}_F)$ .

Indeed, since  $M' \in \mathcal{G}^2(\mathcal{W}_F)$ , there exists a short exact sequence  $0 \rightarrow M' \rightarrow H \rightarrow M'' \rightarrow 0$  in  $R\text{-Mod}$  with  $H \in \mathcal{G}(\mathcal{W}_F)$  and  $M'' \in \mathcal{G}^2(\mathcal{W}_F)$ . Now consider the following push-out diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & K & \longrightarrow & G' \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & H & \longrightarrow & H' & \longrightarrow & G' \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & M'' & \equiv & M'' & & \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

From the middle row of the above diagram, we know  $H' \in \mathcal{G}(\mathcal{W}_F)$  by Corollary 3.6. Then there exists a short exact sequence  $0 \rightarrow H' \rightarrow C \otimes_R F \rightarrow H'' \rightarrow 0$  in  $R\text{-Mod}$  with  $F$  flat and  $H'' \in \mathcal{G}(\mathcal{W}_F)$ . Consider another push-out diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & H' & \longrightarrow & M'' \longrightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \longrightarrow & K & \longrightarrow & C \otimes_R F & \longrightarrow & K' \longrightarrow 0 \\
& & & & \downarrow & & \downarrow \\
& & & & H'' & \xlongequal{\quad} & H'' \\
& & & & \downarrow & & \downarrow \\
& & & & 0 & & 0
\end{array}$$

It is trivial that the third column in the above diagram is as desired. This completes our proof.  $\square$

The following corollary is an immediate consequence of Theorem 3.4 and Theorem 4.5.

**Corollary 4.6.** *Let  $R$  be a GF-closed ring. We have  $\mathcal{G}^n(\mathcal{GF}_C(R) \cap \mathcal{B}_C(R)) = \mathcal{GF}_C(R) \cap \mathcal{B}_C(R)$  for all  $n \geq 1$ .*

Note that the next result on the class of Gorenstein flat modules is of [17, Theorem 4.3] or [3, 1.2] when we set  $C = R$ .

**Corollary 4.7.** *Let  $R$  be a GF-closed ring. We have  $\mathcal{G}^n(\mathcal{F}) = \mathcal{G}(\mathcal{F})$  for all  $n \geq 1$ .*

## REFERENCES

- [1] M. Auslander and M. Bridger, Stable module Theory, Mem. Amer. Math. Soc. 94 (1969).
- [2] D. Bennis, Rings over which the class of Gorenstein flat modules is closed under extensions, Communications in Algebra. 37 (2009): 855C868.
- [3] S. Bouchiba and M. Khaloui, Stability of Gorenstein flat modules, Glasgow Math. J. 54 (2012), 169C175.
- [4] L. W. Christensen and H. Holm, Ascent properties of Auslander categories, Canad. J. Math. 61 (2009): 76C108.
- [5] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, Walter de Gruyter, New York: Spring-Verlag, 2000.
- [6] E. E. Enochs, O. M. G. Jenda, and B. Torrecillas, Gorenstein flat modules, Nanjing Daxue Xuebao Shuxue Bannian Kan. 10 (1993): 1C9.
- [7] E. S. Golod, G-dimension and generalized perfect ideals, Trudy Mat. Inst. Steklov. 165 (1984): 62C66.

- [8] H. B. Foxby, Gorenstein modules and related modules, *Math. Scand.* 31 (1972): 267C284.
- [9] Y. X. Geng and N. Q. Ding,  $\mathcal{W}$ -Gorenstein modules, *J. Algebra.* 325 (2011): 132C146.
- [10] H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra.* 189 (2004): 167C193.
- [11] H. Holm and P. Jøgensen, Semi-dualizing modules and related Gorenstein homological dimensions, *J. Pure Appl. Algebra.* 205 (2006): 423-445.
- [12] H. Holm and D. White, Foxby equivalence over associative rings, *J. Math. Kyoto Univ.* 47 (2007): 781C808.
- [13] C. H. Huang and Z. Y. Huang, Gorenstein syzygy modules, *J. Algebra.* 324 (2010): 3408C3419.
- [14] S. S. Wagstaff, T. Sharif, and D. White, AB-contexts and stability for Gorenstein flat modules with respect to semidualizing modules, *Algebr Represent Theor.* 14 (2011): 403C428.
- [15] S. S. Wagstaff, T. Sharif, and D. White, Stability of Gorenstein categories, *J. London Math. Soc.* 77 (2008): 481C502.
- [16] D. White, Gorenstein projective dimension with respect to a semidualizing module, *J. Commut. Algebra.* 2 (2010): 111-137.
- [17] G. Yang and Z. K. Liu, Stability of Gorenstein flat category, *Glasgow Math. J.* 54 (2012): 177-191.